General Physics: Electromagnetism, Correction 11

Exercise 1 :

Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a non-uniform current density

$$J = \alpha r$$

where α is a constant. Find the magnetic field everywhere.

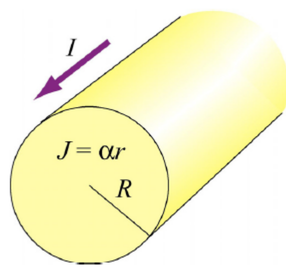


Figure 1: Non-uniform current density

Solution 1 :

To find the magnetic field we apply again the Ampere's law in its full generality

$$\oint_\Gamma d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}} = \mu_0 \int_{\mathcal{A}} d\vec{\sigma} \cdot \vec{J}. \tag{1}$$

We have to distinguish two cases. For $r < R$, we have

$$I_{\text{enc}} = \int_{\mathcal{A}} d\vec{\sigma} \cdot \vec{J} = \int_0^r dr' 2\pi r' (\alpha r') = \frac{2}{3}\pi\alpha r^3. \tag{2}$$

The Ampere's law gives $2\pi r B_1 = \frac{2}{3}\mu_0 \pi \alpha r^3$ and then $B_1 = \alpha\mu_0 r^2/3$.

For $r > R$, we have instead

$$I_{\text{enc}} = \int_{\mathcal{A}} d\vec{\sigma} \cdot \vec{J} = \int_0^R dr' 2\pi r' (\alpha r') = \frac{2}{3}\pi\alpha R^3. \tag{3}$$

The Ampere's law gives $2\pi r B_2 = \frac{2}{3}\mu_0 \pi \alpha R^3$ and then $B_2 = \alpha\mu_0 R^3/3r \propto 1/r$, as in the previous exercise.

Exercise 2 :

A circular loop of radius r_0 rotates with angular speed ω in a fixed magnetic field as shown in the Figure below.

1. Find an expression for the emf induced in the loop.
2. If the magnitude of the magnetic field is 25 μT, the radius of the loop is 1 cm, the resistance of the loop is 25 Ω and the rotation rate ω is 3 rad/s, what is the maximum current in the loop?

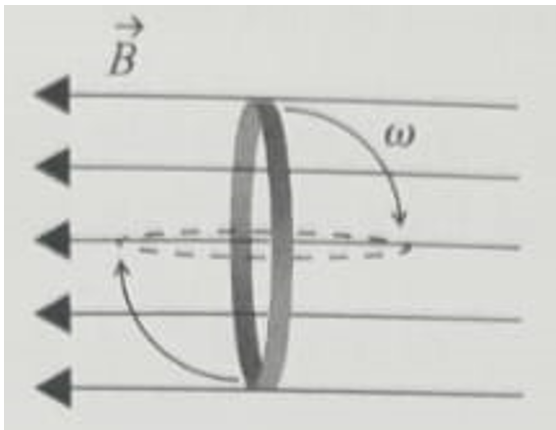


Figure 2: Circular loop rotating in a homogeneous magnetic field at an angular speed ω.

Solution 2 :

1. We can find the emf, ε, from Faraday's law,
$$\varepsilon = -\frac{d\Phi}{dt}, \tag{4}$$
where Φ is the magnetic flux defined as,
$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S}, \tag{5}$$
with $\boldsymbol{B}$ the magnetic field and $\boldsymbol{S}$ the surface vector.
Let's first compute the flux Φ:
$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S} = |\boldsymbol{B}|\, \pi r_0^2 \cos\alpha, \tag{6}$$
$|\boldsymbol{B}|$ denote the absolute value of $\boldsymbol{B}$ and $\alpha = \omega t$ is the angle between magnetic field and the surface vector, we can then write.
$$\Phi = |\boldsymbol{B}|\, \pi r_0^2 \cos\alpha = |\boldsymbol{B}|\, \pi r_0^2 \cos(\omega t). \tag{7}$$

The emf, ε, is then

$$\varepsilon = -\frac{d\Phi}{dt} = -\left|\boldsymbol{B}\right| \pi r_0^2 \frac{d}{dt} \cos\left(\omega t\right) = \left|\boldsymbol{B}\right| \omega \pi r_0^2 \sin\left(\omega t\right). \tag{8}$$

2. We can then find the maximum current I_{max} from Ohm's law.

$$\left|I_{\mathrm{max}}\right| = \frac{\left|\varepsilon_{\mathrm{max}}\right|}{R} = \frac{\left|\boldsymbol{B}\right| \omega \pi r_0^2}{R} = 9.4 \times 10^{-10}\ \mathrm{A}. \tag{9}$$

Exercise 3 :

Figure below shows a long solenoid with radius R and n turns per unit length; its current decreases with time according to

$$I(t) = I_0 e^{-\alpha t}.$$

1. What is the magnitude of the induced electric field at a point a distance r from the central axis of the solenoid when $r > R$ and $r < R$?
2. What is the direction of the induced field at both locations?
3. Sketch the function $E(r)$ for both $r < R$ and $r > R$

   Assume the infinite-solenoid approximation is valid throughout the regions of interest.

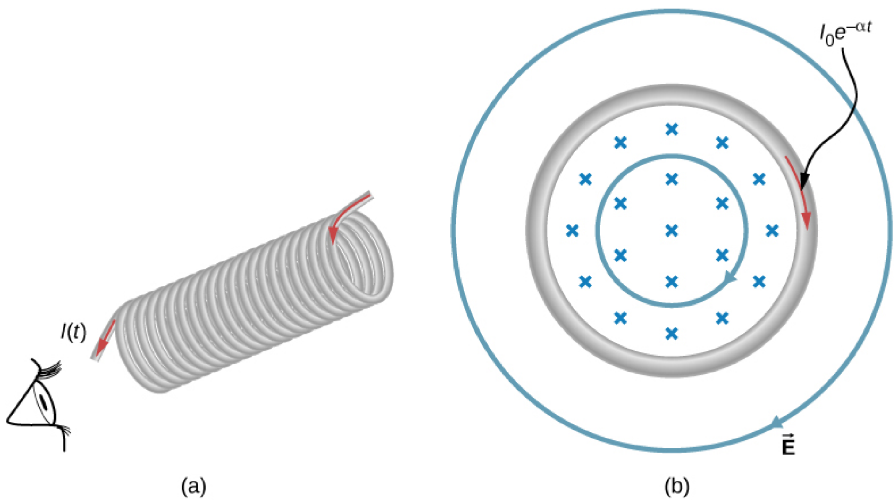


Figure 3: (a) The current in a long solenoid is decreasing exponentially. (b) A cross-sectional view of the solenoid from its left end. The cross-section shown is near the middle of the solenoid. An electric field is induced both inside and outside the solenoid

Solution 3 :

Using the magnetic field of an infinite solenoid and Faraday's law, we compute the induced electric field. Because the system is cylindrically symmetric, the induced electric field is tangent to circular paths and has constant magnitude along each path.

1. The magnetic field inside the solenoid is

   $$B = \mu_0 n I = \mu_0 n I_0 e^{-\alpha t}. \tag{10}$$

   For a circular Amperian loop of radius $r > R$, only the flux within the solenoid contributes. Thus the magnetic flux is

   $$\Phi_m = BA = \mu_0 n I_0 \pi R^2 e^{-\alpha t}. \tag{11}$$

The induced electric field $\vec{E}$ is tangent to the circular loop, so Faraday's law gives

$$\left| \oint \vec{E} \cdot d\vec{\ell} \right| = \left| \frac{d\Phi_m}{dt} \right|. \tag{12}$$

Thus,

$$E(2\pi r) = \left| \frac{d}{dt} \left(\mu_0 n I_0 \pi R^2 e^{-\alpha t} \right) \right| = \alpha \mu_0 n I_0 \pi R^2 e^{-\alpha t}, \tag{13}$$

and therefore, for $r > R$,

$$E = \frac{\alpha \mu_0 n I_0 R^2}{2r} e^{-\alpha t}. \tag{14}$$

The flux linked by a circular path of radius $r < R$ is

$$\Phi_m = B\pi r^2. \tag{15}$$

Using Faraday's law,

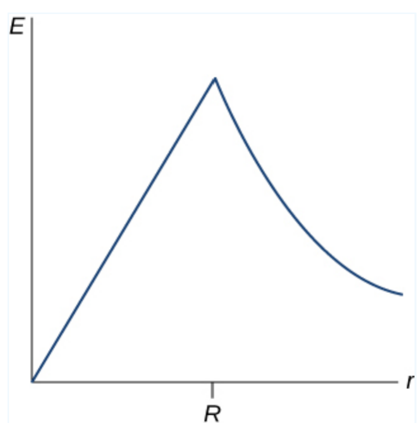
$$E(2\pi r) = \left| \frac{d}{dt} \left(\mu_0 n I_0 \pi r^2 e^{-\alpha t} \right) \right| = \alpha \mu_0 n I_0 \pi r^2 e^{-\alpha t}, \tag{16}$$

so the induced electric field is

$$E = \frac{\alpha \mu_0 n I_0 r}{2} e^{-\alpha t}, \qquad (r < R). \tag{17}$$

2. The magnetic field points into the page (as in the figure) and is decreasing. By Lenz's law, the induced electric field must circulate in the direction that would induce a magnetic field into the page. Therefore, the induced $\vec{E}$ is **counterclockwise** when viewed from the end of the solenoid

3. Inside the solenoid ($r < R$), $|\vec{E}|$ increases linearly with r. Outside the solenoid ($r > R$), the induced electric field decreases as $1/r$.



Exercise 4 :

A small electric car experiences a frictional force of 250 N when moving at a constant speed of 35 km/h. The electric motor is powered by ten batteries of 12 V each, connected in series. It contains a rectangular coil with 270 turns and of size 12×15 cm, which rotates in a magnetic field $B = 0.6$ T. The motor is directly connected (without gearbox) to the wheels with a diameter of 58 cm, so the coil and the wheels turn at the same frequency. We consider the moment when the coil is in the same plane as the magnetic field $\vec{B}$, such that the generated torque is maximal.

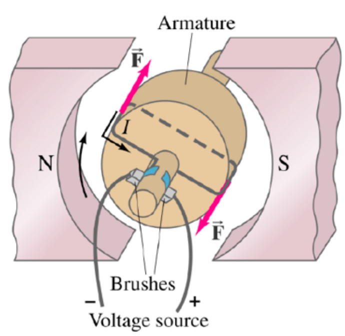
1. What is the current passing through the motor?

   Hint: The mechanical torque must be balanced by the torque generated from the Lorentz force on the coil, given by $\tau_B = \mu B = NIA_{\text{coil}}B$.

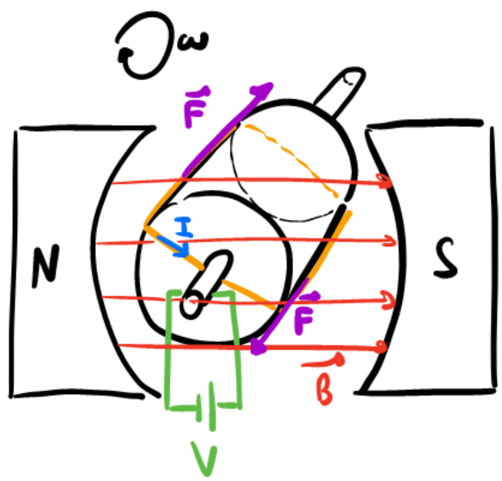
2. What is the *emf* induced in the coil? Plot the magnetic flux $\Phi_B(t)$ through the coil and the induced emf over the time

3. What is the dissipated power in the coil?

4. What percentage of power produced by the motor is delivered to the wheels?



Solution 4 :



1. The coil experiences a Lorentz force due to the current that flows through it and the static external magnetic field $\vec{B}$. Generally speaking, a force acting on a rotating object generates a torque τ according to:
$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{18}$$
The torque generated by the Lorentz force can be written as:
$$\vec{\tau}_B = \vec{\mu} \times \vec{B}, \tag{19}$$
where μ is the magnetic moment of the coil, $\mu = NIA_{\text{coil}}$.

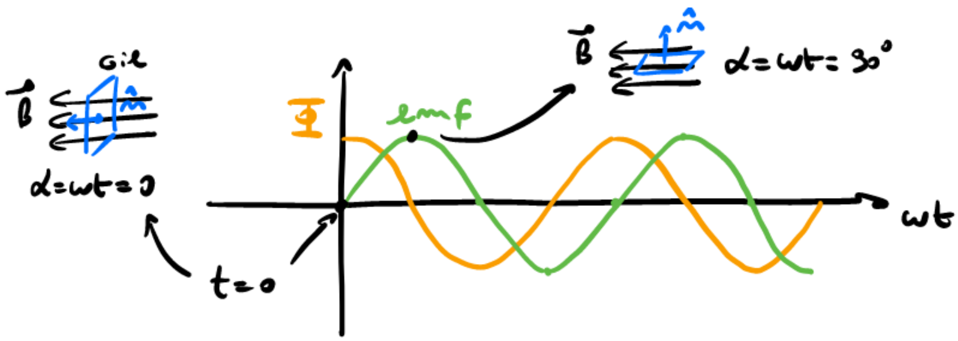
   Since we consider the moment when the coil is in the same plane as the magnetic field B, such that the generated torque is maximal, we have that $\vec{\mu}$ and $\vec{B}$ are perpendicular. The magnitude of the torque is then simply $|\vec{\tau}_B| = \mu B$.

   In addition to the Lorentz force, we have the frictional force F_{fr} that opposes the rotation of the wheel. The torque associated to the frictional force is $\vec{\tau}_{\text{fr}} = \vec{r} \times \vec{F}_{\text{fr}}$. Its magnitude is $|\vec{\tau}_{\text{fr}}| = F_{\text{fr}} d_{\text{wheel}}/2$, being d_{wheel} the diameter of the wheel.

   Since the rotation speed is constant, the two torques must equilibrate, so that $|\vec{\tau}_{\text{fr}}| = |\vec{\tau}_B|$. From this equation we can deduce the current passing through the motor
$$I = \frac{F_{\text{fr}} d_{\text{wheel}}}{2NA_{\text{coil}}B} = 24.9\ A. \tag{20}$$

2. To compute the *emf* induced in the coil we use Faraday's law. Magnetic flux Φ_B is given by $\vec{B} \cdot \vec{A}_{\text{coil}}$ (we discarded the surface integral in the definition of magnetic flux since $\vec{B}$ is uniform). Therefore, calling ω the rotation speed, we have $\Phi_B = BA_{\text{coil}} \cos(\omega t)$. The *emf* then reads
$$emf = -\frac{\partial \Phi_B}{\partial t} = BA_{\text{coil}} \omega \sin(\omega t). \tag{21}$$

We notice that $\Phi_B \propto \cos(\omega t)$ while $emf \propto \sin(\omega t)$, so flux and emf are out-of-phase of 90 degrees.



- At t=0, Φ_B is maximal and the emf is zero.

- At $t = \frac{\pi}{2}$, Φ_B is zero and the emf is maximal:

$$emf_{\text{max}} = BNA_{\text{coil}}\omega = BNA_{coil}\frac{v}{d_{\text{wheel}}/2} = 2BNA_{\text{coil}}\frac{v}{d_{\text{wheel}}} = 97.8\ V, \tag{22}$$

where the fact that $v = \omega r$ has been used.

3. To compute the dissipated power in the coil, we just use the definition of power as $P = IV$. The generated power is given by

$$P_{\text{gen}} = V_{\text{batteries}} \cdot I = 10 \cdot 12 \cdot 24.9 = 2.988\ kW, \tag{23}$$

while the power due to the induced *emf*

$$P_{\text{ind}} = emf \cdot I = 97.8 \cdot 24.9 = 2.435\ kW. \tag{24}$$

Finally, the dissipated power reads

$$P_{\text{diss}} = P_{\text{gen}} - P_{\text{ind}} = 553\ W. \tag{25}$$

4. The percentage of power produced by the motor and delivered to the wheels is given by

$$\eta = \frac{P_{\text{ind}}}{P_{\text{gen}}} = \frac{2.435}{2.988} = 81\%. \tag{26}$$

Exercise 5 :

A square conductive loop with side length $l = 10\,\mathrm{cm}$ and total resistance $R = 10\,\Omega$ is immersed in a uniform magnetic field $B = 0.52\,\mathrm{T}$. The loop is constrained to rotate about a fixed axis orthogonal to the direction of the field, as shown in the figure below. Let θ be the angle between the direction of the magnetic field and the normal to the loop. The loop is kept in rotation with a constant angular velocity $\omega = 12.4\,\mathrm{rad/s}$.

Determine:

1. the absolute value of the electric charge passing through the loop during half a turn between the positions $\theta = 0$ and $\theta = \pi$;

   Hint: In uniform circular motion, a full turn satisfies $2\pi = \omega T$, so the period is $T = 2\pi/\omega$.

2. the external mechanical torque required to maintain the loop in rotation, and its average value over one turn;

   Hint: Remember that the average value over one full period means taking the quantity over a complete cycle and dividing by the duration of that cycle

3. the work done by this mechanical torque over one turn. Remember that the work done by a torque τ is given by $W = \int_{\theta_1}^{\theta_2} \tau d\theta$, where θ_1 and θ_2 represents the initial and final angular positions.

4. the energy dissipated in the loop over one turn.

   Hint: Over one turn means average!!!

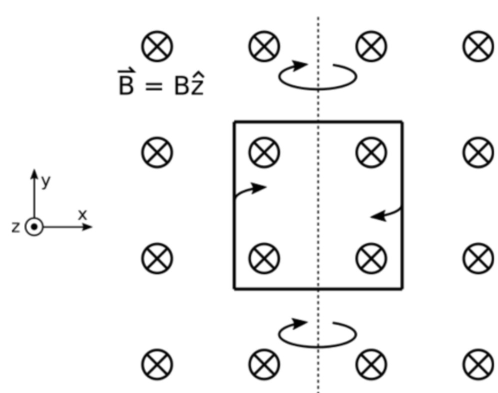


Figure 4: Square conducting loop in magnetic field

Solution 5 :

1. The induced electromotive force (emf) due to the magnetic field acts on the rotating loop. The current circulating in the loop varies with time and is given by Faraday-Neumann-Lenz's law. Assuming that at time $t = 0$ the loop is in the position $\theta = 0$, we have:

$$I(t) = \frac{1}{R}\frac{d\Phi_B}{dt} = -\frac{1}{R}\frac{d}{dt}\left(Bl^2 \cos\theta\right) = \frac{Bl^2\omega}{R}\sin\omega t. \tag{27}$$

   This current, during the first half-turn, flows counterclockwise when viewed from the side of the loop's normal. To determine the value of the charge $Q_{1/2}$ passing through the loop during the first half-turn, we integrate the current over time from $t = 0$ (corresponding to $\theta = 0$) to $t = \pi/\omega$ (corresponding to $\theta = \pi$):

$$Q_{1/2} = \int_0^{\pi/\omega} I(t)\,dt = \frac{Bl^2\omega}{R}\int_0^{\pi/\omega}\sin\omega t\,dt = \frac{2Bl^2}{R} = 1.04\,\text{mC}. \tag{28}$$

2. To maintain the motion at a constant angular velocity, an external mechanical torque is needed to counteract the resistive torque due to the magnetic forces acting on the sides of the loop. This resistive torque is always directed opposite to the direction of the angular velocity vector ω.

   We calculate the magnitude of the resistive torque by equating the resistive moment to the product of the forces acting on the vertical sides of the loop and multiplying by the lever arm:

$$|M(t)| = |M_B(t)| = |I(t)Bl^2\sin\omega t| = \frac{B^2l^4\omega}{R}\sin^2\omega t. \tag{29}$$

   As seen, this value is always positive. To determine the average torque over one turn, we integrate the function $M(t)$ over $t = 0$ to $t = T = 2\pi/\omega$ and divide by the period T. Thus:

$$\overline{M} = \frac{1}{T}\int_0^T M(t)\,dt = \frac{B^2l^4\omega^2}{2\pi R}\int_0^{2\pi/\omega}\sin^2\omega t\,dt = \frac{B^2l^4\omega}{2R} = 1.7\times10^{-5}\,\text{Nm}. \tag{30}$$

3. The work done by the external torque over one turn can be obtained by expressing the torque as a function of the angle θ and integrating over θ from $\theta = 0$ to $\theta = 2\pi$:

$$L_M = \int_0^{2\pi} M(\theta)\,d\theta = \frac{B^2l^4\omega}{R}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{\pi B^2l^4\omega}{R} = 1.05\times10^{-3}\,\text{J}. \tag{31}$$

   It can be easily noted that the same result is obtained by simply multiplying 2π by the average torque.

4. To obtain the energy dissipated in the loop over one turn, we calculate the power dissipated by the Joule effect:

$$E_{\text{diss}} = \int_0^T I^2(t)R\,dt = \frac{B^4l^4\omega^2}{R^2}\int_0^T\sin^2\omega t\,dt = \frac{\pi B^2l^4\omega}{R} = 1.05\times10^{-3}\,\text{J}. \tag{32}$$

As expected, the same result as obtained above corresponds to the work done by the external mechanical torque. All the work done by the torque is dissipated due to the Joule effect.